Stability of an Asymmetric Dual-Spin Spacecraft with Flexible Platform

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This paper investigates the variational stability of a general dual-spin spacecraft consisting of an asymmetric rigid rotor and a flexible platform. Two cases of rotors are considered: constant spin-rate rotors and axially torque-free rotors. The full equations of motion are derived and conditions for the existence of a pure spin motion, referred to as the desired solution, are obtained. A linearization about this solution yields equations with periodic coefficients. A special class of spacecraft is observed, in which the linearized equations are the same for both cases of rotors. Two stability-analysis methods are developed and used for stability analysis of the linearized system under consideration: a numerical scheme based on the Floquet theory and Hsu's single-pass method, and a perturbation method based on the multiple-scales method. A numerical example is presented of a platform consisting of a rigid core and a beam-like antenna extended along the spin axis of the rotor. The effects of the rotor and platform asymmetry, rotor location, and platform damping are demonstrated and explained.

Introduction

THE term dual-spin spacecraft was first used by Likins[1] to describe a spacecraft consisting of two bodies, designed to spin about a common axis of rotation. The first spacecraft of this kind, OSO-1 (Orbiting Solar Observatory), was launched in 1962.

There exists vast literature on the stability of dual-spin spacecraft. However, only few papers deal with the case where both bodies of the spacecraft are asymmetric. In this case, the linearized equations of motion involve periodic coefficients and closed-form solutions are not known. Tsuchiya[2] investigated the attitude motion of a dual-spin spacecraft composed of two slightly asymmetric rigid bodies by the method of averaging. Lukich and Mingori[3] addressed the same problem, without the limitation of small asymmetries, by two numerical methods, the Hill's infinite determinant method and the N pass Floquet method.

Agrawal[4] investigated the stability of a dual-spin spacecraft consisting of two slightly asymmetric bodies connected by a slightly asymmetric, slightly dissipative flexible joint by a perturbation method. In order to obtain the common form of a nongyroscopic system, Agrawal used a first-order approximation before the expansion, and thus eliminated the possibility of considering higher-order terms in the asymptotic expansion. Boundaries of stability were obtained by using the equations developed by Hsu[5,6] for nongyroscopic multidegree of freedom systems.

Tsuchiya[2] and Haixing[7] investigated the nonstationary motion of a completely rigid dual-spin spacecraft through the unstable region generated by the asymmetry of the rotor. They showed that a despinning of a platform through this unstable region might be problematic in case the despinning motor is weak.

The present study extends the work of Refs. 2–4 to the general case of a force-free dual-spin spacecraft consisting of a flexible body, called the platform, and a rigid body, asymmetric with respect to the spin axis, called the rotor. The platform may include dissipative flexible elements of continuous or discrete type. Unlike Refs. 2 and 4, there is no restriction on the size of the platform asymmetry, and for one of the methods used here, there are also no restrictions on the size of the rotor asymmetry and the amount of the platform damping.

Regarding the rotor, two cases are considered; a constant speed rotor and an axially torque free rotor. Following Refs. 8 and 9, a spacecraft of the first case is called a Kelvin gyrostat, and a spacecraft of the second spacecraft is called an apparent gyrostat.

This problem is important, since, practically speaking, asymmetry is unavoidable, and in some cases, where the rotor is large and contains many parts, it may also be desirable in order to simplify the manufacturing of this rotor and to reduce its mass.

Equations of Motion

The idealized system under consideration is shown in Fig. 1. The system consists of two main bodies: the platform designated by P, and the rotor designated by R. The platform may include rigid elements and flexible elements of continuous or discrete types, while the rotor is rigid. The $OXYZ$ frame (Cartesian coordinate system) is an inertial frame, the $ox^Ry^Rz^R$ frame is attached to the rigid rotor, and the $oxyx$ frame is attached to some points in the platform or to a reference configuration of the platform. In the numerical example investigated later, the configuration of the platform at the pure spin motion is chosen as a reference. The following is assumed:

1) The rotor is dynamically and statically balanced with respect to its spin axis. That is, the spin axis of the rotor is a principal axis of the rotor.

2) the $ox^Ry^Rz^R$ frame is attached to the principal axes of the rigid rotor, such that the z^R axis coincides with the spin axis of the rotor. A^R, B^R, and C^R are the principal moments of inertia about the x^R, y^R, and z^R axes.

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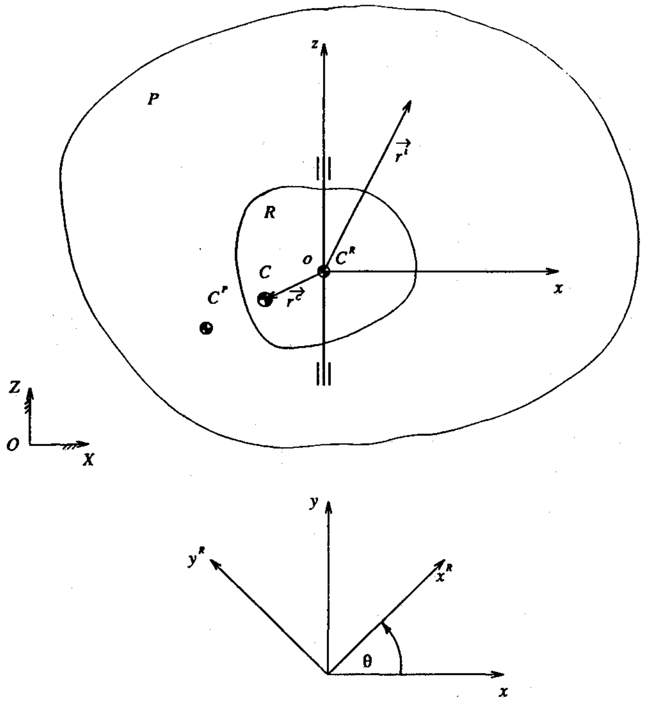

Fig. 1 Model of the system.

3) The rotor is asymmetric with respect to z^R axis, i.e., $A^R \neq B^R$.

4) The platform has n degrees of freedom with respect to the $oxyz$ frame. That is, position vector r^i of any point in the platform with respect to the $oxyz$ frame may be defined by a transformation equation of the form $r^i = r^i\,(r^{i0}, q)$, where r^{i0} is a position vector of this point in the platform at the reference configuration. Both vectors r^{i0} and r^i are expressed in the $oxyz$ frame. The column matrix q contains the unconstrained internal generalized coordinates, $q_1, \ldots, q_n$. It is assumed that the transformation equations do not depend explicitly on time. In case the platform also contains continuous flexible elements, it is assumed that a discretization is performed before the derivation.

5) Any deformation of the platform may be associated with a change in a potential energy function of the form $V = V(q_1, \ldots, q_n)$. The potential energy function and its first and second partial derivatives with respect to q_j are assumed to be continuous in the range of interest.

6) Any deformation of the platform may be associated with energy dissipation. That is, there might exist internal generalized forces Q_j such that $Q_j \dot{q}_j \leq 0$ for $j = 1, \ldots, n$.

7) The system is free from external forces.

8) The internal generalized coordinate θ defines the position of the rotor with respect to the $oxyz$ frame. For the Kelvin gyrostat, this coordinate is constrained by

$$\theta = \sigma^0 t \tag{1}$$

where σ^0 is the constant spin rate of the rotor. Hence, the system has $(n+7)$ degrees of freedom with respect to an inertial frame for the apparent gyrostat, and $(n+6)$ for the Kelvin gyrostat.

It can be shown that the kinetic energy T of the system is

$$2T \triangleq \int_s \dot{r}^i \cdot \dot{r}^i \,\mathrm{d}m = m^s(\dot{d}_x^2 + \dot{d}_y^2 + \dot{d}_z^2) + \omega \cdot \bar{I}(q,\theta) \cdot \omega$$
$$+ \sum_{j,k} P_{jk}(q)\dot{q}_j\dot{q}_k + 2\omega \cdot \sum_j h_j(q)\dot{q}_j + C^R\dot{\theta}^2 + 2C^R\omega_z\dot{\theta} \tag{2}$$

where all Cartesian vectors that appear in Eq. (2) are expressed in the $oxyz$ frame, d_x, d_y, and d_z are the inertial X, Y, and Z coordinates of the common center of mass, m^s is the total mass of the system, ω is the angular velocity vector of the platform frame, ω_z is the z component of the platform angular velocity vector ω,

$$P_{jk}(q) \triangleq \int_P (r^i(q)_{,qj} \cdot r^i(q)_{,qk})\,\mathrm{d}m - m^s r^c(q)_{,qj} \cdot r^c(q)_{,qk}$$

$$h_j(q) \triangleq \int_P (r^i(q) \times r^i(q)_{,qj})\,\mathrm{d}m - m^s r^c(q) \times r^c(q)_{,qj}$$

and $\bar{I}(q,\theta)$ is the inertia dyadic of the system defined as

$$\bar{I}(q,\theta) \triangleq \bar{I}^P(q) + \bar{I}^R(\theta)$$

where the bar designates a dyadic, and

$$\bar{I}^P(q) \triangleq \int_P (r^i(q)^2\bar{1} - r^i(q)r^i(q))\,\mathrm{d}m$$
$$-m^s\big(r^c(q)^2\bar{1} - r^c(q)r^c(q)\big)$$

$$r^c(q) = 1/m^s \int_P r^i(q)\,\mathrm{d}m$$

$$\bar{I}^R(\theta) \triangleq \int_R \big(r^i(\theta)^2\bar{1} - r^i(\theta)r^i(\theta)\big)\,\mathrm{d}m$$

It can also be shown that the angular momentum of the system is

$$H \triangleq \int_{P,R} r^i \times \dot{r}^i \,\mathrm{d}m = \bar{I}(q,\theta)\cdot\omega + \sum_j h_j(q)\dot{q}_j + C^R\dot{\theta}\hat{z} \tag{3}$$

where C^R is the moment of inertia of the rotor about the z^R axis, and $\hat{z}$ is a unit vector along the z axis.

The equations of motion are derived by a hybrid approach. Three equations are found from the Euler equation:

$$\dot{H} = 0 \tag{4}$$

while the others are obtained from the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\dot{d}_x}) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(T_{,\dot{d}_y}) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}(T_{,\dot{d}_z}) = 0 \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\dot{q}_j}) - T_{,q_j} + V_{,q_j} = Q_j, \qquad j = 1, \ldots, n \tag{6}$$

Another equation is derived for the apparent gyrostat from the Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\dot{\theta}}) - T_{,\theta} = 0 \tag{7}$$

Substitution of Eq. (3) in Eq. (4) yields

$$\bar{I}\cdot\dot{\omega} + C^R\ddot{\theta}\hat{z} + \sum_k h_k\ddot{q}_k = -(\bar{I}^R_{,\theta}\dot{\theta} + \sum_k \bar{I}^P_{,q_k}\dot{q}_k)\cdot\omega$$
$$-\sum_{k,l} h_{k,q_l}\dot{q}_k\dot{q}_l - \omega\times\bar{I}\cdot\omega - \omega\times\sum_k h_k\dot{q}_k - C^R\dot{\theta}\omega\times\hat{z} \tag{8}$$

Equation (5) yields three uncoupled equations from which it is concluded that

$$\ddot{d}_x = \ddot{d}_y = \ddot{d}_z = 0 \tag{9}$$

Substitution of Eq. (2) in Eq. (6) yields

$$h_j\cdot\dot{\omega} + \sum_k P_{jk}\ddot{q}_k = -\omega\cdot\sum_k h_{j,q_k}\dot{q}_k + \omega\cdot\sum_k h_{k,q_j}\dot{q}_k$$
$$-\sum_{k,l} P_{jk,q_l}\dot{q}_k\dot{q}_l + {}^1\!/\!_2\sum_{k,l} P_{kl,q_j}\dot{q}_j\dot{q}_k + {}^1\!/\!_2\,\omega\cdot(\bar{I}^P - \bar{I}^m)_{,q_j}\cdot\omega$$
$$-V_{,q_j} + Q_j, \qquad j = 1, \ldots, n \tag{10}$$

Equations (8) and (10) with the substitution of Eq. (1), besides the three uncoupled equations, Eq. (9), constitute the equations of motion for the Kelvin gyrostat. For the apparent gyrostat, another equation is obtained from Eq. (7):

$$C^R(\dot{\omega}_z + \ddot{\theta}) = \frac{\Delta^R}{2}\left((\omega_x{}^2 - \omega_y{}^2)\sin 2\theta - 2\omega_x\omega_y\cos 2\theta\right) \tag{11}$$

where Δ^R is the rotor asymmetry, defined as $\Delta^R \triangleq (B^R - A^R)$, and ω_x, ω_y, and ω_z are the angular velocity components of the platform frame in this frame.

Equations (8), (10), and (11), for the apparent gyrostat, may be written in matrix form

$$[A]\dot{x} = f \tag{12}$$

where, for the apparent gyrostat, $[A] = [A(x)]$, $f = f(x)$, and $x = x^A \triangleq [\omega_x, \omega_y, \omega_z, \dot{\theta}, \dot{q}_1, \ldots, \dot{q}_n, \theta, q_1, \ldots, q_n]^t$, while for the Kelvin gyrostat, $[A] = [A(x,t)]$, $f = f(x,t)$, and $x = x^k \triangleq [\omega_x, \omega_y, {}_z, \dot{q}_1, \ldots, \dot{q}_n, q_1, \ldots, q_n]^t$. Matrix $[A]$ has the form

$$[A] = \begin{bmatrix} [M] & [0] \\ [0] & [1] \end{bmatrix} \tag{13}$$

where $[M]$ is a symmetric matrix referred as the mass matrix of the system, and, as shown below, it is positive definite.

Although Eq. (12) is autonomous for the apparent gyrostat, it is T^0 periodic for the Kelvin gyrostat, where

$$T^0 = |\pi/\sigma^0| \tag{14}$$

Since the equations of motion are derived by two methods, one has to show that Eqs. (8) to (11) are independent. This is shown below by proving that matrix $[A]$ in Eq. (12) is positive definite, and, therefore, nonsingular. Let, for the apparent gyrostat,

$$z^A \triangleq [\theta, q_1, \ldots, q_n]^t$$

and

$$y^A \triangleq [\omega_x, \omega_y, \omega_z, \dot{\theta}, \dot{q}_1, \ldots, \dot{q}_n]^t$$

and let, for the Kelvin gyrostat,

$$z^K \triangleq [q_1, \ldots, q_n]^t$$

and

$$y^K \triangleq [\omega_x, \omega_y, \omega_z, \dot{q}_1, \ldots, \dot{q}_n]^t$$

then, from Eq. (2), one can show that for the apparent gyrostat

$$2T = m^s(\dot{d}_x^2 + \dot{d}_y^2 + \dot{d}_z^2) + y^{A^t}[M(z^A)]y^A$$

and for the Kelvin gyrostat

$$2T = m^s(\dot{d}_x^2 + \dot{d}_y^2 + \dot{d}_z^2) + y^{K^t}\left[(z^K, t)\right]y^K + C^R\dot{\theta}^2 + 2C^R\omega_z\dot{\theta}$$

Since, as defined in this paper, the kinetic energy is a positive-definite function for all $(\dot{d}_x, \dot{d}_y, \dot{d}_z, y^A)$, and also for $(\dot{d}_x, \dot{d}_y, \dot{d}_z, \dot{\theta}, y^K)$, matrix $[M]$ is positive definite, and as a result, so is matrix $[A]$.

Linearization

Of special interest are solutions of Eq. (12), defined by

$$x^0 = [0, 0, \Omega^P, \sigma^0, 0, \ldots, 0, \sigma^0 t, q_1^0, \ldots, q_n^0]^t \qquad \text{apparent gyrostat} \tag{15a}$$

$$x^0 = [0, 0, \Omega^P, 0, \ldots, 0, q_1^0, \ldots, q_n^0]^t \qquad \text{Kelvin gyrostat} \tag{15b}$$

where Ω^P, σ^0, $q_1^0, \ldots, q_n^0$ are specified constants. This solution, referred as the desired solution in this paper, describes a pure spin motion of the rotor and of the nonvibrating platform about the z axis. The desired solution is a time-varying solution for the apparent gyrostat, and a constant solution for the Kelvin gyrostat. Substitution of Eq. (15) into Eq. (12) yields the following conditions for the existence of such a solution

$$\tfrac{1}{2} I_{zz}^{P^0}{}_{,q_j} \Omega^{P2} = V^0{}_{,q_j}, \qquad j = 1, \ldots, n \tag{16a}$$

$$I_{xz}^0 = I_{yz}^0 = 0 \tag{16b}$$

where superscript $(\ )^0$ denotes a value at the desired solution, and I_{xz} and I_{yz} are components of $\bar{I}$. Equation (16) implies that the z axis, which is also the spin axis of the rotor, must be parallel to a principal axis of the platform at the desired solution. Hence, the $oxyz$ frame may be so chosen such that the x and y axes are also parallel to the other principal axes of the platform at the desired solution.

The linearized equations of motion in the neighborhood of the desired solution x^0 are obtained by expanding Eq. (12) in a Taylor series and ignoring second- and higher-order terms. The result is

$$\delta\dot{x} = \left[J^0(t)\right]\delta x, \qquad \left[J^0(t + T^0)\right] = \left[J^0(t)\right] \tag{17}$$

where $\delta x \triangleq x - x^0$, and $[J^0]$ is the Jacobian matrix of $F \triangleq [A]^{-1}\bar{f}$, i.e., $J_{jk}^0 \triangleq (F_{j,x_k})_{x=x^0}$.

It can be shown that, for the desired solution x^0 as defined in Eq. (15),

$$\left[J^0(t)\right] = \left[A^0(t)\right]^{-1}\left[B^0(t)\right]$$

where

$$\left[A^0(t)\right] \triangleq [A]_{x=x^0}, \qquad \left[A^0(t + T^0)\right] = \left[A^0(t)\right]$$

and

$$B_{jk}^0(t) \triangleq (f_{j,x_k})_{x=x^0}, \qquad \left[B^0(t + T^0)\right] = \left[B^0(t)\right]$$

Another form of Eq. (17) is

$$\left[A^0(t)\right]\delta\dot{x} = \left[B^0(t)\right]\delta x \tag{18}$$

For a special class of spacecraft, referred here as reducible spacecraft,

$$h_{zj}^0 = I_{zz}^{Pm0}{}_{,q_j} \equiv 0, \qquad j = 1, \ldots, n \tag{19}$$

Substitution of Eq. (19) into Eq. (17) yields

$$\delta\omega_z = \text{const} = \delta\omega_z(t = 0) \qquad \text{apparent and Kelvin gyrostat}$$

$$\delta\dot{\theta} = \text{const} = \delta\dot{\theta}(t = 0) \qquad \text{apparent gyrostat}$$

In this case, $\delta\omega_z$, $\delta\dot{\theta}$, and $\delta\theta$ may be eliminated from the linearized apparent gyrostat system reducing it by three equations, and $\delta\omega_z$ may be eliminated from the linearized Kelvin gyrostat system, reducing it by one equation. Hence, for this class of spacecraft, the apparent and Kelvin gyrostats share the same reduced linearized equations of the form

$$\left[A^r(t)\right]\delta\dot{x}^r = \left[B^r(t)\right]\delta x^r, \qquad \left[A^r(t + T^0)\right] = \left[A^r(t)\right]$$

$$\left[B^r(t + T^0)\right] = \left[B^r(t)\right] \tag{20}$$

or of the form

$$\delta \dot{x}^r = [J^r(t)]\delta x^r, \qquad [J^r(t+T^0)] = [J^r(t)] \tag{21}$$

where

$$x^r \triangleq [\delta\omega_x, \delta\omega_y, \delta\dot{q}_1, \ldots, \delta\dot{q}_n, \delta q_1, \ldots, \delta q_n]^t$$

and

$$[J^r(t)] \triangleq [A^r(t)]^{-1}[B^r(t)]$$

Figure 2 shows platforms of reducible and nonreducible spacecraft with point-mass dampers, free to move along the z axis.

The linearized equations may be normalized by introducing three characteristic quantities:

1) B^*: a characteristic moment of inertia, defined as $B^* \triangleq \frac{1}{2}(A^P + A^R + B^P + B^R)$, where A^P and B^P are the principal transverse moments of inertia I^P_{xx} and I^P_{yy} at the desired solution.

2) t^*: a characteristic time scale, defined as $t^* \triangleq B^*/(C^P\Omega^P + C^R\Omega^R)$, where $\Omega^R \triangleq \Omega^P + \sigma^0$ and C^P is the principal moment of inertia I^P_{zz} at the desired solution. It can be shown that $(1/t^*)$ is equal to the inertial natural frequency of a completely rigid asixymmetric dual-spin spacecraft.

3) q^*: a characteristic length.

This process leads to the normalized variables

$$\hat{\omega}_x \triangleq \delta\omega_x t^*, \qquad \hat{\omega}_y \triangleq \delta\omega_y t^*, \qquad \hat{\omega}_z \triangleq \delta\omega_z t^*$$

$$\hat{\sigma} \triangleq \delta\dot{\theta} t^*, \qquad \hat{q}_j \triangleq \delta q_j / q^*, \qquad \hat{t} \triangleq t/t^*$$

and, for a reducible system, to the normalized rigid body parameters

$\hat{\Delta}^R$ = the normalized rotor's asymmetry, defined as $\hat{\Delta}^R \triangleq (B^R - A^R)/2B^*$

$\hat{\Delta}^P$ = the normalized platform's asymmetry, defined as $\hat{\Delta}^P \triangleq (B^P - A^P)/2B^*$

$\hat{\Omega}^P$ = the normalized platform's spin rate, defined as $\hat{\Omega}^P \triangleq \Omega^P t^*$

$\hat{\Omega}^R$ = the normalized rotor's spin rate, defined as $\hat{\Omega}^R \triangleq \Omega^R t^*$

For a nonreducible spacecraft, another normalized rigid body parameter is required to define the system, $\hat{C} \triangleq (C^R + C^P)/B^*$.

The period of the normalized system is $\hat{T}^0 = T^0/t^* = \pi/|\hat{\Omega}^P - \hat{\Omega}^R|$. Additional normalized parameters are required in case the platform is flexible.

Without affecting the generality of the investigation, one may assume that $B^R \geq A^R$ and $B^P \geq A^P$. Then, it can be shown that

$$0 \leq \hat{\Delta}^R \leq 1 \tag{22}$$

$$0 \leq \hat{\Delta}^P \leq 1 \tag{23}$$

and

$$0 \leq \hat{\Delta}^P + \hat{\Delta}^R \leq 1 \tag{24}$$

In a gyrostat, a dual-spin spacecraft in which the rotor is large compared to the platform, $\hat{\Delta}^R$ might be large. On the other hand, the rotor in a bias momentum spacecraft is much smaller than the platform, hence only a small $\hat{\Delta}^R$ is expected. The opposite situation is valid for $\hat{\Delta}^P$.

One can show that $\hat{C} \leq 2$ from which it is concluded that that $\hat{\Omega}^P$ and $\hat{\Omega}^R$ cannot have values less than 0.5 at the same time. Usually, a gyrostat starts its mission in an all spun state,[3] where the platform and rotor spin as a single body. After the spacecraft enters the desired orbit, the platform is then despun to an almost zero spin rate.

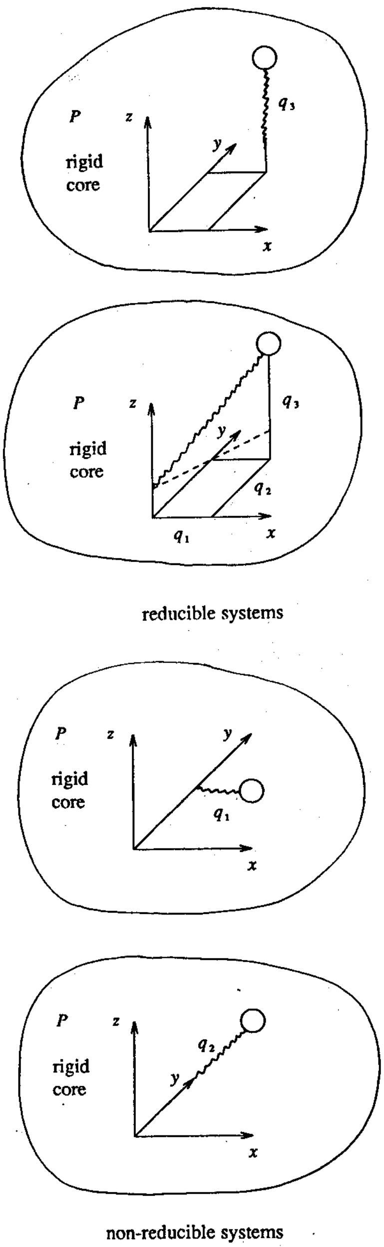

Fig. 2 Platforms of reducible and nonreducible systems.

Stability-Analysis Methods

According to Ref. 10, three approaches for stability analysis of a periodic linear system such as Eq. (17) have been employed in the literature: Hill's method of infinite determinants, perturbation methods, and the Floquet method. The first method is a numerical method by which boundaries of stability may be approximately obtained. According to this reference, this method is numerically inefficient for many degree-of-freedom systems. The second method yields approximate analytical expressions for the boundaries of stability, and therefore, a clear insight may be obtained. However, it is confined to a small rotor asymmetry and small platform damping, or to large period. In the Floquet method, the transition matrix over a period is evaluated numerically, and stability is determined according to values of the eigenvalues of the transition matrix. Since it can deal with any size of parameters, this method is considered as the most general one. However, this method does not provide a clear insight, and it requires substantial computational efforts for evaluating the transition matrix. To overcome the last problem, efficient single-pass numerical schemes were developed for evaluating the transition matrix.[10,11]

As explained below, because of the special form of the system under consideration, available single-pass numerical schemes are not efficient, and available perturbations methods are not appropriate. Hence, two methods are developed below: an efficient single-pass numerical scheme, based on Hsu's numerical scheme,[12,13] and referred as the modified Hsu method, and a perturbation method based on the multiple-scales method.

Modified Hsu Method

Let $[\Phi^T]$ be the transition matrix over period of Eq. (17). That is, $[\Phi^T]$ is a fundamental solution matrix of Eq. (17) at $t = T^0$ generated by the initial condition matrix [1]. The eigenvalues μ_j of $[\Phi^T]$ are called characteristic multipliers of $[J^0]$ [or of the system, Eq. (17)]. The following theorem results from the Floquet-Liapunov theory[14,15]:

The null solution of Eq. (17) is 1) (Liapunov) uniformly asymptotically stable if and only if all of the characteristic multipliers of $[J^0]$ have a modulus less than one; 2) (Liapunov) uniformly stable if and only if no characteristic multiplier has a modulus greater than one, and if all of the characteristic multipliers with a modulus equal to one are associated with Jordan blocks of order one in the Jordan canonical form of $[J^0]$; and 3) (Liapunov) unstable if one or more characteristic multipliers have a modulus greater than one, or if at least one characteristic multiplier with a modulus equal to one is associated with a Jordan block of order greater than one in the Jordan canonical form of $[J^0]$.

In the single-pass approach, an approximate transition matrix $[\Phi^T]$ is obtained by a single integration. For example, in single-step schemes (e.g., Runge-Kutta schemes), the time interval $0 \leq t \leq T^0$ is divided into s time intervals, Δt_j $(j = 1, \dots, s)$, not necessarily the same. Also, let t_j be the time at the end of the jth interval. Then, a state vector at $t = t_j$ may be written as an approximate function of the state vector at $t = t_{j-1}$ in the form of $x_j = [\Phi_j] x_{j-1}$ where $x_j \triangleq x(t = t_j)$, so that

$$x_s = [\Phi_s][\Phi_{s-1}], \dots, [\Phi_1] x_0$$

Hence, an approximate expression for the transition matrix is given by

$$[\Phi^T] = \prod_{j=s}^{1} [\Phi_j]$$

where it is agreed that $[\Phi_j]$ appears to the left of $[\Phi_{j-1}]$.

The single-pass numerical schemes described in the literature[10,11] are numerically inefficient for the system under consideration, since the inverse of matrix $[A^0]$ has to be computed at least once for each step of integration. For example, the single-pass Runge-Kutta-Gill scheme developed in Ref. 10 requires the computation of the inverse of $[A^0]$ at least twice for each step of integration.

Another deficiency of these schemes is that since a minimum number of time intervals is required to avoid overflow or large numbers, accuracy cannot be compromised with CPU time. For example, it is shown in Table 1 of Ref. 10 that overflow or large numbers result for a number of intervals less than 80. However, for a number of intervals greater than 80, the results are very accurate, and there is no difference between 80 and 200 intervals, except for larger CPU time in the second case.

The scheme developed here overcomes these problems due to the following features: 1) the computation of the inverse of $[A^0]$ is not required at each step of integration; 2) when needed, computation the inverse of $[A^0]$ is performed only once; and 3) accuracy can be compromised with CPU time.

The periodic system, Eq. (18), is approximated by a series of systems of equations with constant coefficients of the form

$$[A_j^{av}]\delta\dot{x} = [B_j^{av}]\delta x, \qquad t_j > t \geq t_{j-1} \tag{25}$$

where

$$[A_j^{av}] \triangleq \frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} \left[A^0(t)\right] \mathrm{d}t \tag{26a}$$

$$[B_j^{av}] \triangleq \frac{1}{\Delta t_j} \int_{t_{j-1}}^{t_j} \left[B^0(t)\right] \mathrm{d}t \tag{26b}$$

Hence,

$$[J_j^{av}] \approx [A_j^{av}]^{-1}[B_j^{av}] \tag{27}$$

Since, in our case, matrices $[A^0(t)]$ and $[B^0(t)]$ are composed of constants and simple harmonic elements, therefore Eq. (27) has a simple closed-form solution.

By introducing another modification, one can control the accuracy/CPU time without being limited by overflow or large numbers. This improvement is achieved by decreasing s, the number of time intervals Δt_j in which the transverse of $[A_j^{av}]$ is computed, accompanied by subdivision of each Δt_j into r_j subintervals Δt_{jk} $(k = 1, \dots, r_j)$ in order to avoid overflow or large numbers. From the theory of linear differential equations with constant coefficients,

$$x_{jk} = \exp\left(\Delta t_{jk}[J_j^{av}]\right) x_{jk-1}, \qquad k = 1, \dots, r_j, \qquad j = 1, \dots, s$$

where $x_{jk} \triangleq x(t = t_{j-1} + \sum_{i=1}^{k} \Delta t_{ji})$. Hence, matrix $[\Phi_j]$ is obtained from

$$[\Phi_j] = \prod_{k=r_j}^{1} \exp(\Delta t_{jk}[J_j^{av}]) \tag{28}$$

Multiple-Scales Method

One may write Eq. (17) in the form

$$\delta\dot{x} - [J^{pr}]\delta x = \epsilon\left\{\left(\cos 2\sigma^0 t[L^1] + \sin 2\sigma^0 t[L^2]\right)\delta\dot{x} + (\Omega^P + 2\sigma^0)\left(\cos 2\sigma^0 t[L^2] - \sin 2\sigma^0 t[L^1]\right)\delta x - \frac{1}{\epsilon}[L^3]\delta x\right\} \tag{29}$$

where

$$\epsilon \triangleq \frac{\Delta^R}{2}$$

is assumed to be a small number, $[J^{pr}]$ is the constant part of $[J^0]$,

$$[L^1] \triangleq [A^{pr}]^{-1}\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & -1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

$$[L^2] \triangleq [A^{pr}]^{-1}\begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

and $[L^3] \triangleq [A^{pr}]^{-1}[D]$. $[A^{pr}]$ is the constant part of $[A^0]$ and

$[D]$ is the damping matrix, defined as $D_{jk} = -(Q_{j,x_k})_{x=x^0}$. The left-hand side of Eq. (29) describes an undamped system with an axisymmetric rotor of an averaged transverse moment of inertia equal to $(A^R + B^R)/2$.

Most perturbations methods for multi-degree-of-freedom linear systems that are found in the literature deal with systems that have the form

$$[M]\ddot{x} + [K]x = \epsilon f(\ddot{x}, \dot{x}, x, t) \tag{30}$$

where ϵ is a small parameter, $[M]$ is a symmetric positive definite matrix, $[K]$ is a symmetric matrix, and f is a periodic function with respect to t. Employing a modal decomposition, Eq. (30) may be transformed into the form

$$\ddot{x}_j + w_j^2 x_j = \epsilon \sum_{k=1}^{N} \left(p_{jk}^1(t)\ddot{x}_k + p_{jk}^2(t)\dot{x}_k + p_{jk}^3(t)x_k\right) \quad j = 1, \ldots, N \tag{31}$$

where w_j are natural frequencies of the unperturbed system, and $p_{jk}^i(t + T^0) = p_{jk}^i(t)$. In general,[5,6,16] perturbation methods for such systems start with the form of Eq. (31), which cannot be achieved easily in systems that also involve large gyroscopic terms, (first-order time derivatives in the second-order form of the equations of motion, caused by "Coriolis forces"), as the system under consideration here. Agrawal,[4] who dealt with a linear gyroscopic system of the form

$$[M]\ddot{x} + [G]\dot{x} + [K]x = \epsilon f(\ddot{x}, \dot{x}, x, t) \tag{32}$$

where $[G]$ is a skew symmetric matrix, used a first-order approximation before the expansion in order to obtain the form of Eq. (31), and thus eliminated the possibility of a higher-order consideration. Ariaratnam and Namachchivaya[17] used the averaging method for a gyroscopic system like that of Eq. (32) without invoking a first-order approximation before the expansion. Here, due to the derivation in terms of angular velocities of the platform frame rather than in terms of Euler angles, and in the general case where the platform is asymmetric, the system under consideration, Eq. (29), cannot be cast into the form of Eq. (32). Hence, a different perturbation method is developed and used.

Let λ_j and δx_j be the eigenvalues and eigenvectors of

$$\left(\lambda[1] - [J^{pr}]\right)\delta x = 0 \tag{33}$$

and $[x]$ be a matrix whose columns are $\delta x_1, \ldots, \delta x_N$. Only the case where all of the eigenvectors are independent is dealt here, so that $[x]$ is nonsingular. Then, premultiplying Eq. (29) by $[x]^{-1}$, and substituting $\delta x = [x]u$ yield

$$\dot{u} - [\Lambda]u = \epsilon\Big\{\left(\cos 2\sigma^0 t[\Gamma^1] + \sin 2\sigma^0 t[\Gamma^2]\right)\dot{u} + (\Omega^P + 2\sigma^0) \times \left(\cos 2\sigma^0 t[\Gamma^2] - \sin 2\sigma^0 t[\Gamma^1]\right)u - (1/\epsilon)[\Gamma^3]u\Big\} \tag{34}$$

where $[\Gamma^j] \triangleq [x]^{-1}[L^j][x]$, and $[\Lambda]$ is a diagonal matrix with λ_j in the diagonal.

In index notation, Eq. (34) becomes

$$\dot{u}_j - \lambda_j u_j = \epsilon\left\{\sum_{k=1}^{N} (\cos 2\sigma^0 t\,\Gamma_{jk}^1 + \sin\sigma^0 t\,\Gamma_{jk}^2)\dot{u}_k + (\Omega^P + 2\sigma^0)\sum_{k=1}^{N} (\cos 2\sigma^0 t\,\Gamma_{jk}^2 - \sin 2\sigma^0 t\,\Gamma_{jk}^1)u_k - \frac{1}{\epsilon}\sum_{k=1}^{N} \Gamma_{jk}^3 u_k\right\} \quad j = 1, \ldots, N \tag{35}$$

Following the multiple-scales method, we seek a uniformly valid expansion having the form

$$u_j \approx u_{j0}(\tau_0, \tau_1, \tau_2, \ldots) + \epsilon u_{j1}(\tau_0, \tau_1, \tau_2, \ldots) + (\epsilon)^2 u_{j2}(\tau_0, \tau_1, \tau_2, \ldots) + \cdots \tag{36}$$

where $\tau_j \triangleq t(\epsilon)^j$ so that

$$\frac{d}{dt}(\;) = \frac{\partial}{\partial_{\tau_0}}(\;) + \epsilon\frac{\partial}{\partial_{\tau_1}}(\;) + \cdots \tag{37}$$

and it is assumed that, since ϵ is small, the time scales τ_j are independent. Substituting Eqs. (36) and (37) into Eq. (35), and equating coefficients of the same power of ϵ yield

$$D_0 u_{j0} - \lambda_j u_{j0} = 0, \qquad j = 1, \ldots, N \tag{38}$$

$$D_0 u_{j1} - \lambda_j u_{j1} = -D_1 u_{j0} + \sum_{k=1}^{N} (\cos 2\sigma^0 \tau_0 \Gamma_{jk}^1 + \sin 2\sigma^0 \tau_0 \Gamma_{jk}^2) D_0 u_{k0} + (\Omega^P + 2\sigma^0)\sum_{k=1}^{N} (\cos 2\sigma^0 \tau_0 \Gamma_{jk}^2 - \sin 2\sigma^0 \tau_0 \Gamma_{jk}^1) u_{k0} - \frac{1}{\epsilon}\sum_{k=1}^{N} \Gamma_{jk}^3 u_{k0}, \qquad j = 1, \ldots, N \tag{39}$$

$$D_0 u_{j2} - \lambda_j u_{j2} = \cdots \tag{40}$$

$$\vdots$$

where

$$D_j(\;) \triangleq \frac{\partial}{\partial \tau_j}(\;)$$

Below, necessary conditions for stability are obtained by considering only zero- and first-order terms in the expansion. The general solution of Eq. (38) has the form

$$u_{j0} = \gamma_j(\tau_1, \tau_2, \ldots)e^{\lambda_j \tau_0} \qquad j = 1, \ldots, N \tag{41}$$

where γ_j are complex numbers. Because of the origin of the linearized equations, one may show that if λ_j is an eigenvalue so is $-\lambda_j$. Hence, from Eq. (41), one concludes that a necessary condition for stability of the solution Eq. (36), is that all λ_j must be purely imaginary numbers. Let

$$\Gamma_{jk}^I \triangleq \Gamma_{jk}^1 + i\,\Gamma_{jk}^2 \tag{42a}$$

$$\Gamma_{jk}^{II} \triangleq \Gamma_{jk}^1 - i\,\Gamma_{jk}^2 \tag{42b}$$

Also, by definition

$$\sin 2\sigma^0 \tau_0 \equiv (1/2i)(e^{2i\sigma^0 \tau_0} - e^{-2i\sigma^0 \tau_0}) \tag{43a}$$

$$\cos 2\sigma^0 \tau_0 \equiv \tfrac{1}{2}(e^{2i\sigma^0 \tau_0} + e^{-2i\sigma^0 \tau_0}) \tag{43b}$$

Substitution of Eqs. (41-43) into Eq. (39) yields

$$D_0 u_{j1} - \lambda_j u_{j1} = -D_1 \gamma_j(\tau_1, \tau_2, \ldots)e^{\lambda_j \tau_0} + \sum_{k=1}^{N}\Big\{\tfrac{1}{2}(\lambda_k + i\Omega^P + 2i\sigma^0)\Gamma_{jk}^{II} e^{(\lambda_k + 2i\sigma^0)\tau_0} + \tfrac{1}{2}(\lambda_k - i\Omega^P - 2i\sigma^0)\Gamma_{jk}^I e^{(\lambda_k - 2i\sigma^0)\tau_0} - \frac{1}{\epsilon}\Gamma_{jk}^3 e^{\lambda_k \tau_0}\Big\}\gamma_k(\tau_1, \tau_2, \ldots), \qquad j = 1, \ldots, N \tag{44}$$

In order to obtain a uniformly valid expansion, γ_j must be chosen in such a way to eliminate troublesome terms from u_j^1. Observation of Eq. (44) shows two classes of resonances that must be taken into consideration: combined resonances where

$$\sigma^0 \approx \pm\frac{1}{2i}(\lambda_j - \lambda_k)$$

and simultaneous resonances where

$$\sigma^0 \approx \pm\frac{1}{2i}(\lambda_j - \lambda_k) \approx \pm\frac{1}{2i}(\lambda_l - \lambda_m) \approx \cdots$$

Case a: σ^0 Is Away from Resonances

Recalling that a necessary condition for stability away from resonances is that $\lambda_j = iw_j$, and that if iw_j is an eigenvalue, so is $-iw_j$, then in this case σ^0 is away from all

$$\pm(w_j \pm w_k)/2, \qquad j,k = 1,\ldots,N/2$$

For a valid expansion, we require

$$(D_1 + \Gamma^3_{jj}/\epsilon)\gamma_j(\tau_1,\tau_2,\ldots) = 0, \qquad j = 1,\ldots,N$$

from which

$$\gamma_j(\tau_1,\tau_2,\ldots) = \eta_j(\tau_2,\ldots)e^{-\frac{\Gamma^3_{jj}}{\epsilon}\tau_1}, \qquad j = 1,\ldots,N \tag{45}$$

Hence, another necessary condition for stability away from resonances is that all real parts of Γ^3_{jj} must be positive. Thus, stability of a system with weak asymmetry and damping, away from resonances, may be approximately determined by investigating an equivalent system, in which the rotor is axisymmetric with transverse moment of inertia equal to $(A^R + B^R)/2$.

Case b: $\sigma^0 \approx \pm(1/2i)(\lambda_l - \lambda_m)$

The case $\sigma^0 \approx +(1/2i)(\lambda_l - \lambda_m)$ with $l \neq m$ is first investigated. This case corresponds to $\sigma^0 \approx ½(\pm w_l \pm w_m)$. The case $\sigma^0 \approx \pm ½(w_l + w_m)$ is called a resonance of an addition type, while the case where $\sigma^0 \approx \pm ½(w_l - w_m)$ is called a resonance of a difference type.

For u_{j1} $(j \neq l,m)$, troublesome terms are eliminated from Eq. (44), by requiring

$$\left(D_1 + \frac{\Gamma^3_{jj}}{\epsilon}\right)\gamma_j(\tau_1,\tau_2,\ldots) = 0, \quad j = 1,\ldots,N, \quad j \neq l,m \tag{46}$$

from which it is concluded that a necessary condition for stability is that

$$\mathrm{Re}(\Gamma^3_{jj}) \geq 0, \qquad j = 1,\ldots,N, \qquad j \neq l,m$$

For the solutions of u_{l1} and u_{m1}, a detuning parameter σ^1 is introduced

$$\sigma^1 \triangleq \frac{1}{i\epsilon}\left(2i\sigma^0 - (\lambda_l - \lambda_m)\right) \tag{47}$$

where σ^1 is a real number in case λ_l and λ_m are purely imaginary. Hence, for $j = l$, Eq. (44) becomes

$$\begin{aligned}(D_0 - \lambda_l)u_{l1} = &-D_1\gamma_l(\tau_1,\tau_2,\ldots)e^{\lambda_l\tau_0}\\ &+ ½\sum_{k\neq m}(\lambda_k + i\Omega^P + 2i\sigma^0)\Gamma^{II}_{lk}e^{(\lambda_k+2i\sigma^0)\tau_0}\gamma_k(\tau_1,\tau_2,\ldots)\\ &+ ½\sum_k(\lambda_k - i\Omega^P - 2i\sigma^0)\Gamma^I_{lk}e^{(\lambda_k-2i\sigma^0)\tau_0}\gamma_k(\tau_1,\tau_2,\ldots)\\ &+ ½(\lambda_m + i\Omega^P + 2i\sigma^0)\Gamma^{II}_{lm}e^{(\lambda_l\tau_0 + i\sigma^1\tau_1)}\gamma_m(\tau_1,\tau_2,\ldots)\\ &- \frac{1}{\epsilon}\sum_k \Gamma^3_{lk}e^{\lambda_k\tau_0}\gamma_k(\tau_1,\tau_2,\ldots)\end{aligned} \tag{48}$$

from which it is recognized that troublesome terms are eliminated by requiring

$$D_1\gamma_l - ½(\lambda_m + i\Omega^P + 2i\sigma^0)\Gamma^{II}_{lm}e^{i\sigma^1\tau_1}\gamma_m + \frac{1}{\epsilon}\Gamma^3_{ll}\gamma_l = 0 \tag{49}$$

Similarly, troublesome terms are eliminated from $u_{m1}(\tau_1)$ by requiring

$$D_1\gamma_m - ½(\lambda_l - i\Omega^P - 2i\sigma^0)\Gamma^I_{ml}e^{-i\sigma^1\tau_1}\gamma_l + \frac{1}{\epsilon}\Gamma^3_{mm}\gamma_m = 0 \tag{50}$$

Elimination of γ_l from Eqs. (49) and (50) yields

$$\begin{aligned}\{&(D_1)^2 + D_1(\Gamma^3_{ll}/\epsilon + \Gamma^3_{mm}/\epsilon + i\sigma^1) + i\sigma^1\Gamma^3_{mm}\epsilon\\ &+ \Gamma^3_{ll}\Gamma^3_{mm}/(\epsilon)^2 - ¼\Xi_{lm}\}\gamma_m = 0\end{aligned}$$

where

$$\begin{aligned}\Xi_{lm} &\triangleq (\lambda_l - i\Omega^P - 2i\sigma^0)(\lambda_m + i\Omega^P + 2i\sigma^0)\Gamma^I_{ml}\Gamma^{II}_{lm}\\ &= (\lambda_m - i\Omega^P - i\epsilon\sigma^1)(\lambda_l + i\Omega^P + i\epsilon\sigma^1)\Gamma^I_{ml}\Gamma^{II}_{lm}\end{aligned} \tag{51}$$

Hence, a necessary condition for stability is that the two characteristic values of

$$\begin{aligned}&(v)^2 + v(\Gamma^3_{ll}/\epsilon + \Gamma^3_{mm}/\epsilon + i\sigma^1) + i\sigma^1\Gamma^3_{mm}\epsilon\\ &+ \Gamma^3_{ll}\Gamma^3_{mm}/(\epsilon)^2 - ¼\Xi_{lm} = 0\end{aligned} \tag{52}$$

must have negative or zero real parts. In case Γ^3_{ll}, Γ^3_{mm} and Ξ_{lm} are real, one may show that this condition is

$$(\sigma^1)^2 \geq (\Gamma^3_{ll} + \Gamma^3_{mm})^2\left(\frac{\Xi_{lm}}{4\Gamma^3_{ll}\Gamma^3_{mm}} - \frac{1}{(\epsilon)^2}\right) \tag{53}$$

Substitution of Eq. (51) into Eq. (53) yields the final expression for σ^1 on the boundary of stability:

$$\begin{aligned}&(\sigma^1)^2\left(\xi_{lm} - (\epsilon)^2\Gamma^{II}_{lm}\Gamma^I_{ml}\right) + i\epsilon\sigma^1\Gamma^{II}_{lm}\Gamma^I_{ml}(\lambda_l - \lambda_m + 2i\Omega^P)\\ &+ 4\Gamma^3_{ll}\Gamma^3_{mm}/(\epsilon)^2 - (\lambda_l + i\Omega^P)(\lambda_m - i\Omega^P)\Gamma^{II}_{lm}\Gamma^I_{ml} = 0\end{aligned} \tag{54}$$

where

$$\xi_{lm} \triangleq \frac{4\Gamma^3_{ll}\Gamma^3_{mm}}{(\Gamma^3_{ll} + \Gamma^3_{mm})^2}$$

and $\xi_{lm} = 1$ in case $\Gamma^3_{ll} = \Gamma^3_{mm} = 0$. Examination of Eq. (54) reveals that the influence of the platform damping decreases as the rotor asymmetry increases, and, to a first-order approximation and positive ξ_{lm}, two boundaries of stability exist for this resonance, if and only if

$$\epsilon > 2\left(\frac{\Gamma^3_{ll}\Gamma^3_{mm}}{(\lambda_l + i\Omega^P)(\lambda_m - i\Omega^P)\Gamma^{II}_{lm}\Gamma^I_{ml}}\right)^{½} \tag{55}$$

One may also show from Eq. (54) that damping decreases the unstable region if and only if

$$(i\Omega^P - \lambda_m)(i\Omega^P + \lambda_l)\Gamma^I_{ml}\Gamma^{II}_{lm} < \left(\frac{4\Gamma^3_l\Gamma^3_m}{\epsilon(\Gamma^3_l - \Gamma^3_m)}\right)^2 \tag{56}$$

For the case $\sigma^0 \approx -(1/2i)(\lambda_l - \lambda_m)$, we have $\sigma^0 \approx (1/2i)(\lambda_m - \lambda_l)$. Hence, by interchanging indices l and m, Eqs. (53) and (54) are applicable also for this case.

Case c: $\sigma^0 \approx \frac{1}{2i}(\lambda_j - \lambda_k) \approx \frac{1}{2i}(\lambda_l - \lambda_m)$, $l \neq j,k,m$, $m \neq j,k,l$

Two detuning parameters, σ^1 and σ^2, are introduced for this type of uncoupled simultaneous resonances:

$$\sigma^1 \triangleq \frac{1}{i\epsilon}\left(2i\sigma^0 - (\lambda_j - \lambda_k)\right), \qquad \sigma^2 \triangleq \frac{1}{i\epsilon}\left(2i\sigma^0 - (\lambda_l - \lambda_m)\right) \tag{57}$$

Similar to the previous case, we obtain now two equations for the boundary:

$$\begin{aligned}&(\sigma^1)^2\left(\xi_{jk} - (\epsilon)^2\Gamma^{II}_{jk}\Gamma^I_{kj}\right) + i\epsilon\sigma^1\Gamma^{II}_{jk}\Gamma^I_{kj}(\lambda_j - \lambda_k + 2i\Omega^P)\\ &+ 4\Gamma^3_{jj}\Gamma^3_{kk}/(\epsilon)^2 - (\lambda_j + i\Omega^P)(\lambda_k - i\Omega^P)\Gamma^{II}_{jk}\Gamma^I_{kj} = 0\end{aligned} \tag{58}$$

$$(\sigma^2)^2\big(\xi_{lm}-(\epsilon)^2\Gamma^{II}_{lm}\Gamma^{I}_{ml}\big)+i\epsilon\sigma^1\Gamma^{II}_{lm}\Gamma^{I}_{ml}(\lambda_l-\lambda_m+2i\Omega^P)$$
$$+4\Gamma^3_{ll}\Gamma^3_{mm}/(\epsilon)^2-(\lambda_l+i\Omega^P)(\lambda_m-i\Omega^P)\Gamma^{II}_{lm}\Gamma^{I}_{ml}=0 \qquad (59)$$

Noting that Eq. (58) originates from the requirement for stability of u_{j1} and u_{k1}, and Eq. (59) originates from the requirement for stability of u_{l1} and u_{m1}, one concludes that the actual boundary, the one that determines the boundary of the whole system, is the one that includes the largest unstable region.

This conclusion is also valid for the case where there are more than two simultaneous uncoupled resonances.

Case d: $\sigma^0\approx\frac{1}{2i}(\lambda_j-\lambda_k)\approx\frac{1}{2i}(\lambda_j-\lambda_l),\quad k\neq l,\quad j\neq k,l$

In this case of simultaneous coupled resonances, two detuning parameters are introduced:

$$\sigma^1\triangleq\frac{1}{i\epsilon}\big(2i\sigma^0-(\lambda_j-\lambda_k)\big),\quad \sigma^2\triangleq\frac{1}{i\epsilon}\big(2i\sigma^0-(\lambda_j-\lambda_l)\big) \qquad (60)$$

The characteristic polynomial for this case is

$$(v)^3+(v)^2(\Gamma^3_{jj}/\epsilon+\Gamma^3_{ll}/\epsilon+\Gamma^3_{kk}/\epsilon-i\sigma^1-i\sigma^2)+v/(\epsilon)^2$$
$$\times\{\Gamma^3_{kk}\Gamma^3_{ll}+\Gamma^3_{jj}(\Gamma^3_{kk}-i\epsilon\sigma^1)+\Gamma^3_{jj}(\Gamma^3_{ll}-i\epsilon\sigma^2)-(\epsilon)^2$$
$$\times\sigma^1\sigma^2-i\epsilon\Gamma^3_{kk}\sigma^2-i\epsilon\Gamma^3_{ll}\sigma^1-\tfrac14(\epsilon)^2\Xi_{jk}-\tfrac14(\epsilon)^2\Xi_{jl}\}$$
$$-\tfrac14\Xi_{jk}(\Gamma^3_{ll}/\epsilon-i\sigma^2)-\tfrac14\Xi_{jl}(\Gamma^3_{kk}/\epsilon-i\sigma^1)+\Gamma^3_{jj}/(\epsilon)^3$$
$$\times\big(\Gamma^3_{kk}\Gamma^3_{ll}-i\epsilon\Gamma^3_{ll}\sigma^1-i\epsilon\Gamma^3_{kk}\sigma^2-(\epsilon)^2\sigma^1\sigma^2\big)=0 \qquad (61)$$

For stability, all real parts of the characteristic values of Eq. (61) must be negative or zero. Noting from Eqs. (51) and (60) that Ξ_{jl}, Ξ_{jk}, σ^1, and σ^2 are functions of σ^0, then one may conclude that the task of obtaining a closed-form expression for σ^0 on the boundary is complicated for this case. In a numerical approach, one has to determine stability by checking each real part of the characteristic values of Eq. (61) for each σ^0.

Numerical Example

This section includes a variational stability analysis of dual-spin spacecraft in which the platform is composed of a rigid core and a rectangular cross-sectional beam-like antenna extended along the spin axis of the rotor. A schematic diagram of a platform of such a system is shown in Fig. 3.

It is assumed that 1) the antenna is made of homogeneous isotropic material and it is uniform in the z direction; 2) the planes of the beam at rest are parallel to the xz and yz planes, as shown in Fig. 3; 3) the common center of mass at rest

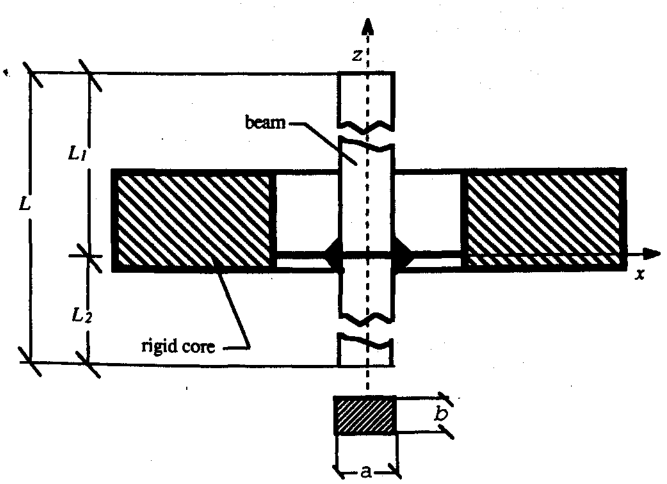

Fig. 3 Platform composed of a rigid core and a beam-like antenna.

coincides with the center of mass of the rotor; 4) the centerline of the beam at rest coincides with the rotor's spin axis; and 5) each branch of the antenna is clamped to the rigid part of the platform.

Because of the assumptions 3 and 4 the system is a reducible one. The antenna is modeled according to Euler-Bernoulli theory of beams, and a discretization before the derivation is done according to the assumed mode method. The deflection of the neutral axis in the xy plane is approximated by

$$x(z,t)\approx\sum_{j=1}^{m}\phi^r_j(z)q^{xr}_j(t),\qquad r=1,2 \qquad (62)$$

$$y(z,t)\approx\sum_{j=1}^{m}\phi^r_j(z)q^{yr}_j(t),\qquad r=1,2 \qquad (63)$$

where $q^{sr}_j(t)$ are the internal generalized coordinates. Superscript s designates the direction of deflection: $s=x$ for a deflection in the xz plane and $s=y$ for a deflection in the yz plane. Superscript r designates the branches of the antenna, $r=1$ for the upper branch and $r=2$ for the lower branch. Here, the $\phi^r_j(z)$ are the modal functions of a clamped-free beam, i.e.,

$$\phi^r_j(z)=\cosh\left(\beta_j\frac{z}{L_r}\right)-\cos\left(\beta_j\frac{z}{L_r}\right)+(-1)^r\sigma_j$$
$$\times\left[\sinh\left(\beta_j\frac{z}{L_r}\right)-\sin\left(\beta_j\frac{z}{L^r}\right)\right]\qquad j=1,\dots,m,\ r=1,2 \qquad (64)$$

where β_j and σ_j are real constants that can be found in textbooks, dealing with bending vibrations of beams.

The total length of the beam L is chosen as a characteristic length. Ten normalized parameters are required to define the system, four of them are the rigid-body parameters described above: $\hat{\Omega}^P$, $\hat{\Omega}^R$, $\hat{\Delta}^P$, and $\hat{\Delta}^R$ and the others are $\hat{v}$, $\hat{m}^1$, $\hat{m}^a$, $\hat{\kappa}^a$, $\hat{w}^a$ and $\hat{\zeta}^a$, defined as

$$\hat{v}\triangleq a/b,\qquad \infty\geq\hat{v}\geq 0$$

$$\hat{m}^1\triangleq\frac{L_1}{L},\qquad 1\geq\hat{m}^1\geq 0$$

$$\hat{m}^a\triangleq\frac{\rho abL}{m^s},\qquad 1\geq\hat{m}^a\geq 0$$

$$\hat{\kappa}^a\triangleq\frac{\rho ab(L)^3}{12B*},\qquad \hat{\kappa}^a\geq 0$$

where ρ is the mass density of the antenna, a and b are the x and y dimensions of a cross section of the antenna. $\hat{w}^a$ is the lowest normalized natural frequency of the whole antenna in the yz plane as a free-free beam, defined as

$$\hat{w}^a\triangleq b\left(\frac{E}{12\rho}\right)^{1/2}\left(\frac{4.73004\dots}{L}\right)^2t*$$

where E is Young's modulus. $\hat{\zeta}^a$ is a normalized damping coefficient, such that the damping coefficients are assumed to be

$$c_{x^q_j y^r_k}\equiv 0,\qquad q,r=1,2,\qquad j,k=1,\dots,m$$

$$c_{x^q_j x^r_k}=\delta_{jk}\delta_{qr}2\hat{\zeta}^a b(a)^2\frac{(\beta_j)^2}{L_r}\left(\frac{\rho E}{12}\right)^{1/2}$$
$$q,r=1,2,\qquad j,k=1,\dots,m$$

$$c_{y^q_j y^r_k}=\delta_{jk}\delta_{qr}2\hat{\zeta}^a a(b)^2\frac{(\beta_j)^2}{L_r}\left(\frac{\rho E}{12}\right)^{1/2}$$
$$q,r=1,2,\qquad j,k=1,\dots,m$$

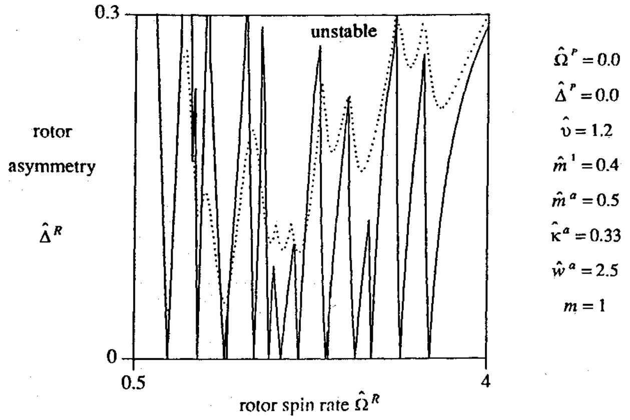

Fig. 4 Stability boundaries for an asymmetric antenna (obtained by the modified Hsu method). (The solid line represents boundaries for $\hat{\zeta}^a = 0.00$ and the dotted line for $\hat{\zeta}^a = 0.10$.)

where

$$c_{x^q_j y^r_k}$$

is a damping coefficient associated with a generalized force Q^{xq}_j and a generalized coordinate q^{yr}_k. It can be shown that $c_{\underline{y}^r\underline{y}^r} = 2\hat{\zeta}^a(\rho abL_r) \times$ (jth natural frequency of the rth clamped-free beam in the sz plane).

Figure 4 shows the unstable regions on the $\hat{\Delta}^R - \hat{\Omega}^R$ plane for a despun platform, obtained by the modified Hsu method. The platform in this example is asymmetric as indicated by the value of $\hat{\upsilon} \neq 1$, and the two branches of the antenna do not have the same length as indicated by the value of $\hat{m}^1 \neq 0.5$. Only one modal function was taken into consideration in Eqs. (62) and (63). Hence the number of equations is ten. In this figure, a boundary was searched at 351 points along the $\hat{\Omega}^R$ axis. All computations were done on the University of Toronto CRAY X-MP/22 supercomputer.

Figure 4 shows 14 partially overlapping unstable regions due to first-order parametric resonances of addition type of the five natural frequencies of the system. Another unstable region, also generated by a parametric resonance of addition type, cannot be seen because of the coarse resolution of this figure.

Effect of Damping

As explained previously, and shown in Fig. 4, three phenomena are observed when damping is introduced: 1) the unstable regions do not touch the $\hat{\Omega}^R$ axis as indicated by Eq. (55); 2) the effect of damping is negligible for large rotor asymmetry; and 3) damping may increase the unstable regions, as indicated by Eq. (56).

Axisymmetric Platform

It can be shown that, in case the platform is axisymmetric, the eigenvalue problem, Eq. (33), may be written in a complex form:

$$\left(\lambda[A^{cr}] - [B^{cr}]\right)\delta\boldsymbol{x}^{cr} = \boldsymbol{0} \tag{65}$$

where $[A^{cr}]$ and $[B^{cr}]$ are complex matrices, and

$$\delta\boldsymbol{x}^{cr} \triangleq \Big[(\omega_x + i\omega_y),\ (\dot{q}_{\underline{1}}^{x1} + i\dot{q}_{\underline{1}}^{y1}),\ (\dot{q}_{\underline{1}}^{x2} + i\dot{q}_{\underline{1}}^{y2}), \ldots$$
$$(q_{\underline{1}}^{x1} + iq_{\underline{1}}^{y1}),\ (q_{\underline{1}}^{x2} + iq_{\underline{1}}^{y2}), \ldots,\ (q_{\underline{m}}^{x1} + iq_{\underline{m}}^{y1}),\ (q_{\underline{m}}^{x2} + iq_{\underline{m}}^{y2})\Big]^t$$

The dimension of Eq. (65) is only half of the dimension of Eq. (33). Hence, a pair of eigenvalues $(\pm iw_j)$ of Eq. (33) coincides to one eigenvalue, $(+iw_j)$ or $(-iw_j)$ of the equivalent problem, Eq. (65). Hence, the eigenvalue $(+iw_j)$ with its associated eigenvectors δx^{cr}_j describe a forward precession, while the eigenvalue $(-iw_j)$ and its associated eigenvector describe a backward precession.

Equivalent to Eq. (29), one may analyze the stability analysis of this system, starting with the perturbation equation in the complex form:

$$\delta\dot{x}^{cr} - [J^{cr}]\delta x^{cr} = \epsilon\Big\{e^{i2\sigma^0 t}[L^{1cr}]\big(\delta\dot{\underline{x}}^{cr} + i(\Omega^P + 2\sigma^0)\delta\underline{x}^{cr}\big)$$
$$-(1/\epsilon)[L^{3cr}]\delta x^{cr}\Big\} \tag{66}$$

where $\delta\underline{x}^{cr}$ is the complex conjugate of δx^{cr}, $[L^{3cr}]$ is the damping matrix, $[J^{cr}] \triangleq [A^{cr}]^{-1}[B^{cr}]$, and

$$[L^{1cr}] \triangleq [A^{cr}]^{-1}\begin{bmatrix} 1 & [0] \\ [0] & [0] \end{bmatrix}$$

Doing so, one concludes that the combined resonances of Eq. (66) are

$$\sigma^0 = \frac{1}{2i}(\lambda_j + \lambda_k), \quad j,k = 1,\ldots,(4m+1), \quad j \neq k \tag{67}$$

where λ_j and λ_k are eigenvalues of $[J^{cr}]$. Hence, for $\sigma^0 > 0$, two eigenvalues that belong to a backward precession do not generate a combined resonance. For this example, one can show that a pair of eigenvectors δx_k and $\delta\underline{x}_k$ describe a forward precession if and only if

$$i(\underline{q}_j^{xr} q_j^{yr} - q_j^{xr}\underline{q}_j^{yr}) > 0$$

where q^{xr}_j and q^{yr}_j are elements of δx_k.

For the spacecraft with axisymmetric platform analyzed in Fig. 5, w_1, w_3, and w_5 are associated with forward precession, while w_2 and w_4 are associated with backward precession. Hence, there are only nine possibilities of parametric resonances for this case: w_1, $(w_1 + w_3)/2$, $(w_1 + w_5)/2$, w_3, $(w_3 + w_5)/2$, w_5, $(w_5 - w_4)/2$, $(w_5 - w_2)/2$, and $(w_3 - w_2)/2$. In Fig. 5, only the six combined resonances of the addition type contribute to unstable regions.

One should note that since the platform is axisymmetric, the periodic coefficients may be eliminated from the problem by writing the equations of motion using a frame attached to the rotor rather than to the platform.

A comparison between the modified Hsu method and the multiple-scales method shows significant differences only in the large range of $\hat{\Delta}^R$.

Symmetric Platform

When the systems of Figs. 4 and 5 are modified by moving the rotor to the middle of the beam, that is, $\hat{m}^1$ is changed from 0.4 to 0.5, the eigenvectors of $[J^{pr}]$ have three basic

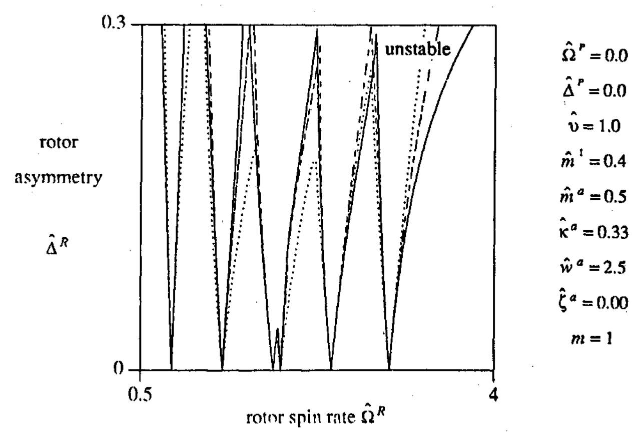

Fig. 5 Stability boundaries for an axisymmetric antenna. [The solid line represents boundaries obtained by the modified Hsu method, the dotted line boundaries according to Eq. (54), and the dashed line boundaries according to Eq. (61).]

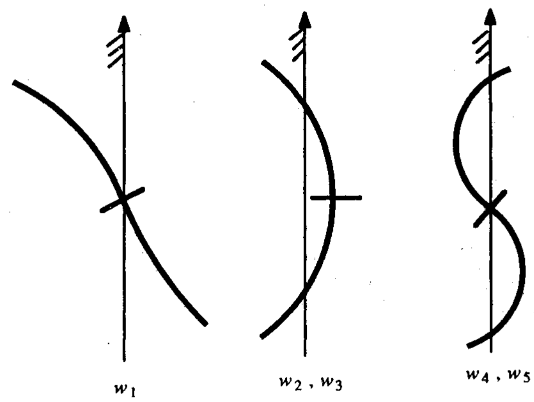

Fig. 6 Basic mode shapes for a symmetric spacecraft with a beam-like antenna.

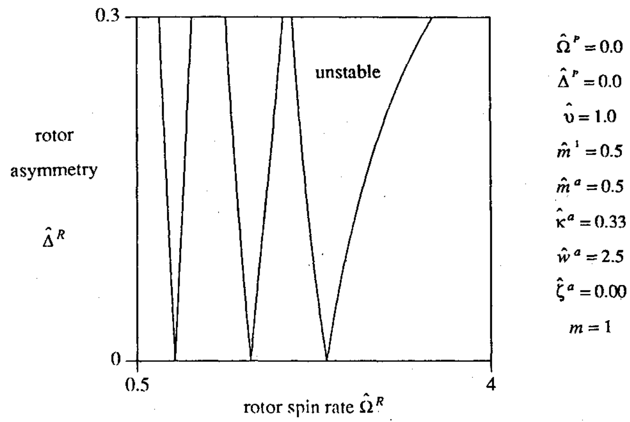

Fig. 7 Stability boundaries for an axisymmetric platform with a symmetric antenna (obtained by the modified Hsu method).

mode shapes, as shown in Fig. 6. The mode shapes associated with natural frequencies w_2 and w_3 are characterized by $\omega_x = \omega_y \equiv 0$. That is, the two eigenvalues w_2 and w_3 with the associated eigenvectors describe a motion in which the rotor moves parallel to itself, and therefore it is expected that these modes are not excited by the asymmetric rotor. This phenomenon is explained below by the multiple-scales method.

Eigenvectors associated with the eigenvalues $(\pm iw_2)$ and $(\pm iw_3)$ of this system are characterized in that their first two elements, corresponding to ω_x and ω_y, are identically zero. Here, such eigenvectors will be called passive modes.

By the definition of $[\Gamma^1]$, $[\Gamma^2]$, $[L^1]$, and $[L^2]$, one may show that

$$\Gamma^i_{jk} = \sum_{l}^{N} \left\{ x^{\text{inv}}_{jl} L^i_{l1} x_{1k} + x^{\text{inv}}_{jl} L^i_{l2} x_{2k} \right\}$$
$$j,k = 1,\ldots,N, \qquad i = 1,2$$

where x_{mk} are elements of the eigenvector matrix $[x]$, and x^{inv}_{jl} are elements of $[x]^{-1}$. Noting that x_{1k} and x_{2k} stand for ω_x and ω_y of the kth eigenvector, and that for a passive mode δx_k, $x_{1k} = x_{2k} = 0$ for all $k = 1,\ldots,N$, one concludes that $\Gamma^{ii}_{jk} = 0$, for all j and i. Hence, from Eq. (35), passive modes are not excited by an asymmetric rotor.

In this case, the location of the rotor is an antinode (a point of maximum deflection) for all symmetric modes. Hence, these modes are passive, and as a result, they are not excited by the asymmetric rotor. Figure 7 shows unstable regions for a system where the rotor is central and the platform is axisymmetric. In this example, w_2 is associated with a passive mode and a backward precession, w_3 is associated with a passive mode, and w_4 is associated with a backward precession. Hence, only w_1 and w_5 generate unstable regions.

Conclusions

Two methods were developed and used in this paper to investigate the variational stability of a dual-spin satellite with asymmetric rotor and a flexible asymmetric platform: a numerical method based on Floquet theory and Hsu's numerical scheme, and a perturbation method, based on the multiple-scales method. It was found that asymmetry in the rotor is undesirable since it may cause instability due to parametric resonances. An axisymmetric platform and a rotor located at an antinode reduce the unstable regions, while damping in the platform may increase or decrease the unstable regions. These two methods may also be used to determine the variational stability of other systems having periodic coefficients in their mass matrix, such as flexible asymmetric rotors on heavy asymmetric bearings.

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